Number of Irreducible Representations

$\lambda^1, \dots, \lambda^s$ - complete list of pairwise inequivalent irreducible G-reps

$\chi_k := \chi_{\lambda^k}$

Theorem: $\chi_1, \dots, \chi_s$ is an orthonormal basis of $L^c(G)$

$\Rightarrow \quad s = \dim(L^c(G)) =$ number of conjugacy classes in G

Lemma: $\varphi : G \to GL(V)$ repr.

$f \in L^c(G)$

Define $T_f = T_f^\varphi : V \to V$ by

$$T_f := \frac{1}{|G|} \sum_{g \in G} \overline{f(g)}\, \varphi_g$$

Then $T_f^\varphi \in \mathrm{Hom}_G(\varphi, \varphi)$

Proof:

$$T_f = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)}\, \varphi_g$$

Show: $\forall\, a \in G \implies \varphi_a T_f \varphi_{a^{-1}} = T_f$

$$\varphi_a T_f \varphi_{a^{-1}} = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)}\, \varphi_{aga^{-1}}$$

$h = aga^{-1}$

$$= \frac{1}{|G|} \sum_{h \in G} \overline{f(a^{-1}ha)}\, \varphi_h$$

$$= \frac{1}{|G|} \sum_{h \in G} \overline{f(h)}\, \varphi_h = T_f.$$

Cor: φ irreducible.

then $T_f^{\varphi} = \lambda I$, $\lambda = \frac{1}{d}\langle \chi, f \rangle$

$\chi = \chi_\varphi$, $d = \dim \varphi$.

Proof: $T_f^{\varphi} = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)}\, \varphi_g \in \mathrm{Hom}_G(\varphi, \varphi)$

φ irred, Schur $\Rightarrow$ $T_f^{\varphi} = \lambda I$

Trace:

$$\mathrm{Tr}(T_f) = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)}\, \chi(g) = \langle \chi, f \rangle$$

$$\mathrm{Tr}(\lambda I) = \lambda d \quad \checkmark$$

Proof of Theorem Show irred char $\chi_1, .., \chi_s$ span $L^c(G)$

$f \in L^c(G)$

$\rightarrow$ $f' := f - \sum_{k=1}^{s} \langle \chi_k, f \rangle \chi_k$. WTS $f' = 0$

$\Rightarrow f \in \mathrm{Span}(\chi_1, .., \chi_s)$

Note: $\langle \chi_k, f' \rangle = 0$

Show: $f \in L^c(G)$, $\langle \chi_k, f \rangle = 0 \ \forall k$

$\Rightarrow f = 0$

$\varphi: G \to GL(V)$ any representation

$f \in L^c(G) \rightsquigarrow T_f^{\varphi} = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)}\, \varphi_g .$

$\varphi = \varphi^{(1)} \oplus \cdots \oplus \varphi^{(r)}$, $\varphi^{(k)}$ irrep

$$T_f^{\varphi}\Big|_{\varphi^{(k)}} = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)}\, \varphi_g^{(k)} = T_f^{\varphi^{(k)}}$$

which $= \lambda I$, $\lambda = \frac{1}{\dim \varphi^{(k)}} \langle \chi_{\varphi^{(k)}}, f \rangle$

Assume. $\langle \chi_{\varphi^{(k)}}, f \rangle = 0$

$\Rightarrow T_f^{\varphi} = 0$ all representations φ.

Plug in: $\varphi = L$ regular rep

$V = \mathbb{C}G$, $\{u_g\}_{g \in G}$, $\dim V = |G|$

Evaluate $T_f^L(u_e)$:

$$\underset{0}{\underset{\parallel}{T_f^L(u_e)}} = \frac{1}{|G|} \sum_{g \in G} \overline{f(g)}\, u_g \Rightarrow \overline{f(g)} = 0$$

$\Rightarrow f \equiv 0$